High order asymptotic for periodic multilayers

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Abstract

In this paper, we propose a rigorous transfer matrix formalism to analyze periodic multilayers in some high-frequency homogenization regime. We derive explicit expressions for transfer matrix when the wavelength is comparable to the size of the periodic cell. We numerically validate our approach by comparing the dispersion law and transmission spectrum (fresnel coefficients) of a stack alternating two dielectric layers against that of an effective bi-anisotropic medium.

1. Introduction

In the last decade, there has been a surge of interest in homogenization of the metamaterials [1, 2], which are artificial materials engineered (i.e.periodic structures) to have desired properties that cannot be found in nature, such as an effective refractive index below unity, or even negative [3, 4]. The classical analytic homogenization theories aim to assign electromagnetic parameters to those artificial materials, but they are typically valid under the limitations that the periodic of the cell at sub-wavelength scales. The extension of classical homogenization theory to high frequencies is of pressing importance for physicists working in the field of photonic crystals, but applied mathematicians also show a keen interest in this topic [5]. In this paper, we show that one can approximate a periodic multilayers by an effective bi-anisotropic medium up to the optical band, using the high-order homogenization [6].

2. Method

A periodic multilayers and its effective bi-anisotropic medium are illustrated in Fig.1. The multilayers is made of n identical unit cells consisting of two layers, both of which are non-magnetic. Let $h = h_1 + h_2$ be the thickness of a unit cell, the layers labelled by $m = 2p + 1$ (p is an integer) have thickness h_1 ($f_1 = h_1/h$) and permittivity ε_1, while the layers labelled by $m = 2p$ have thickness h_2 ($f_2 = h_2/h$) and permittivity ε_2. In [6], we have derived the characteristic matrix and denoted it by $\hat{\omega} M_m$. For each

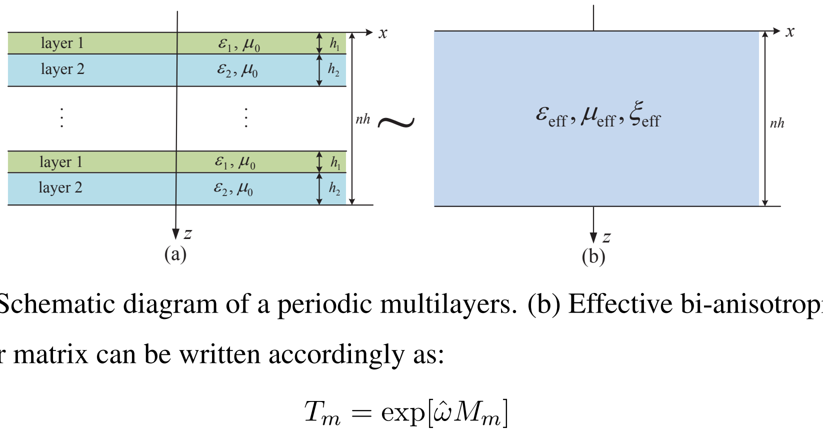

Fig. 1: (a) Schematic diagram of a periodic multilayers. (b) Effective bi-anisotropic medium

layer, the transfer matrix can be written accordingly as:

$$T_m = \exp[\hat{\omega} M_m] \tag{1}$$

with $\hat{\omega} = \omega h/(2\pi c)$ the normalized frequency, which is a small parameter. The transfer matrix for a unit cell is:

$$T = \exp[\hat{\omega} M_2] \exp[\hat{\omega} M_1] \tag{2}$$

The key point of our homogenization algorithm is to derive the expressions of the matrix $T\, (= \exp[\hat{\omega} M_{\mathrm{eff}}])$, when an effective bi-anisotropic medium is defined to approximate the periodic multilayered stack. To carry out the asymptotic analysis, we use the Baker-Campbell-Hausdorff formula (BCH, [7]):

$$\exp\left[A\right] \exp\left[B\right] = \exp\left[A + B + [\![A, B]\!] + [\![A - B, [\![A, B]\!]]\!]/3 + \cdots\right] \tag{3}$$

where the matrix $A + B$ is defined as the first order approximation (classical homogenization), the commutator of A and B ($[\![A, B]\!] = (AB-BA)/2$) is the second order approximation, $[\![A-B, [\![A, B]\!]]\!]/3$ is the third order approximation, and so forth.

Altogether, the transfer matrix T can be written as:

$$T(\hat{\omega}) = I + \hat{\omega} T^{(1)} + \hat{\omega}^2 T^{(2)} + \hat{\omega}^3 T^{(3)} + \cdots \tag{4}$$

3. Results

Using the asymptotic algorithm described in the previous section, we find that the effective bi-anisotropic medium which we mentioned in the introduction is described by the following effective permittivity, permeability and chirality matrices:

$$\varepsilon_{\mathrm{eff}} = \begin{bmatrix} \varepsilon_{\mathrm{p}} & 0 & 0 \\ 0 & \varepsilon_{\mathrm{s}} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{bmatrix}, \quad \mu_{\mathrm{eff}} = \begin{bmatrix} \mu_{\mathrm{s}} & 0 & 0 \\ 0 & \mu_{\mathrm{p}} & 0 \\ 0 & 0 & \mu_{\perp} \end{bmatrix}, \quad \xi_{\mathrm{eff}} = \begin{bmatrix} 0 & \xi_{\mathrm{p}} & 0 \\ \xi_{\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

where the 8 components with subscripts represent the effective parameters in s and p polarization. From (3), the characteristic matrix M_{eff} for the effective medium can be derived. The entries (e.g. 8 components for an oblique incidence) in the effective tensors can also be identified [6].

For the sake of illustration, let us consider the simple case of a normal incident wave in s-polarization. We take up to eighth order approximation in BCH (3) and derive the approximate transfer matrix T with corrective terms $\hat{\omega} T^{(m)}$ with m ranging from 1 to 8, and leading order term $T^{(0)} = I$. It is enlightening to look at the dispersion law, which, from (4), is related to the trace of T in the following way:

$$\cos(k_z h) = \frac{1}{2} tr(T) = 1 - \frac{\hat{\omega}^2}{2}(\varepsilon_1 f_1 + \varepsilon_2 f_2) - \frac{\hat{\omega}^4}{24} f_1^2 f_2^2 (\varepsilon_1 - \varepsilon_2)^2 + \frac{\hat{\omega}^4}{24}(\varepsilon_1 f_1 + \varepsilon_2 f_2)^2 + \cdots \tag{6}$$

with k_z the wavenumber along z-direction. As is known to us, when $|tr(T)/2| \geq 1$, a stop-band is present [8]. Fig. 2(a) shows the dispersion law of a typical effective bi-anisotropic medium at different orders of approximations against that of a classical periodic multilayered stack. Here we suppose the two dielectric layers are respectively Glass ($\varepsilon_1 = 2$, $f_1 = 0.8$), and Silicon ($\varepsilon_2 = 12$, $f_2 = 0.2$).

In Fig. 2(a), comparing with the dispersion law of the multilayered stack (the solid line), we can clearly see that the second, fourth order approximations are just efficient in a range of low frequency, while at the sixth order, the dispersion curves are nearly superimposed up to the edge of the first stop band (with an error of 1% exemplified in the close up view, see inset of Fig. 2(a)), so well beyond the range of validity of classical homogenization [9]. Adding the next even order (i.e. eighth order, the thin solid line), changes the curvature and gives a sharper estimate in the stop band region (a full superimposition as seen in inset), although its intersection with the horizontal axis defines an approximate position for the upper edge of the stop band. This can be improved by taking higher orders of approximation.

In order to check the validity of our homogenization approach, we also plot the transmission coefficient of the homogenized bi-anisotropic medium (dotted line in Fig. 2(b)) with formula:

$$t = \frac{2k_{z0}/\omega\mu_0}{(k_{z0}/\omega\mu_0)(T_{11} + T_{22}) + i(T_{21} - (k_{z0}/\omega\mu_0)^2 T_{12})} \tag{7}$$

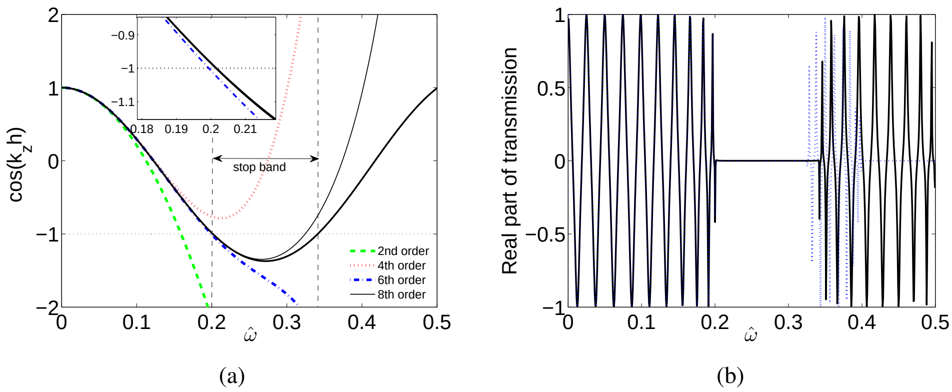

Fig. 2: (a) Dispersion curves for the periodic multilayers (thick solid line) and the effective bi-anisotropic medium in different orders approximation for a normal incidence. Note that p and s polarizations share the same curves; (b) Transmission spectrum for both the multilayered stack (solid line) and the effective bi-anisotropic medium in eighth order approximation (dotted line).

where k_{z0} is the wavenumber in vacuum, and and similarly the formula can be retrievable for an incident wave in p-polarization. It is noted that T_{11} (T_{22}), T_{12} and T_{21} are related to the effective parameters like ξ_{eff}, μ_{eff} and ε_{eff} in the bi-anisotropic medium. And we compare this coefficient against that of the periodic structure (solid line in Fig. 2(b)). We note that the transmission vanishes in the same range of frequencies, up to a slight inaccuracy, which can be reduced by taking higher order asymptotic in (4).

4. Conclusion

In conclusion, we have demonstrated by numerical calculation that an effective bi-anisotropic medium shares the same transmission spectrum as a one-dimensional periodic dielectric structure, when the size of the periodic cell is on the same order as the wavelength. Although we just discussed the case when the incident wave is normal to the stack, we would like to stress that our homogenization algorithm is applicable when there is an oblique incident wave, and also at higher orders of approximation, in which case additional stop bands would be also retrieved from the analysis. Finally, the present analysis can be extended to the layered structures of which the individual layers are inhomogeneous in the transverse plane(two-dimensional periodic structures), even three-dimensional periodic structures.

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