Resonant mode analysis of metamaterials composed of conducting spheres and wires by equivalent circuit model

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Abstract

This paper proposes a systematic method for deriving equivalent circuit models of metamaterials. The equivalent circuit model is efficient to analyze the electromagnetic phenomena in metamaterials and to design them. The target metamaterials we discuss are composed of conducting spheres and wires. Based on the electric field integral equations of Maxwell equations, we derive a lumped constant equivalent circuit model by Galerkin method. The proposed model expresses spheres with a capacitance matrix and wires with an inductance matrix. It clarifies the electromagnetic phenomena in metamaterials such as their resonant modes and their frequencies.

1. Introduction

The modeling of electromagnetic phenomena in metamaterials is very important for analysis and design of them. Equivalent electric circuits of metamaterials have been an efficient model to grasp the electromagnetic phenomena in meta-atoms such as split-ring resonators. In those cases, LC resonant circuits are derived by intuitive way. However, in order to model more complicated structures such as the spatial coupling among the meta-atoms, more systematic approach is required.

This paper proposes an equivalent circuit model of metamaterials which are composed of conducting spheres connected by wires [1]. The proposed model is based on a partial equivalent electric circuit [2], and enables to model the spatial coupling among conductors from Maxwell equations. Using this model, we can analyze systematically the resonant modes and their frequencies of the metamaterials.

2. Equivalent circuit model with spatial coupling

An example of a conductor system composed of conducting spheres and wires is shown in Fig.1(a). Cubic unit cells, whose size are D and smaller than the wavelength λ, are arrayed regularly in vacuum, and conducting spheres connected by wires are at the arbitrary vertex of the unit cells. The number of spheres and wires are M and N, respectively. As shown in Fig.1(b), the radius a of the wire is much smaller than the length l and the radius b of the spheres, and the radius b is much smaller than the length D. In order to obtain an equivalent circuit with spatial coupling of the structure, we start from Maxwell equations in Lorentz gauge. Assuming that the spheres and wires are perfect conductor, we describe the electric field in the conductors by

$$\mu_0 \frac{\partial}{\partial t} \int\int G(\boldsymbol{r},t,\boldsymbol{r}',t')\boldsymbol{J}(\boldsymbol{r}',t')\mathrm{d}^3\boldsymbol{r}'\mathrm{d}t' + \frac{1}{\varepsilon_0}\nabla \int\int G(\boldsymbol{r},t,\boldsymbol{r}',t')\rho(\boldsymbol{r}',t')\mathrm{d}^3\boldsymbol{r}'\mathrm{d}t' = \boldsymbol{E}^E(\boldsymbol{r},t), \quad (1)$$

where $\boldsymbol{J}$, ρ and $\boldsymbol{E}^E$ are induced current density, charge density, and external incident field, respectively.

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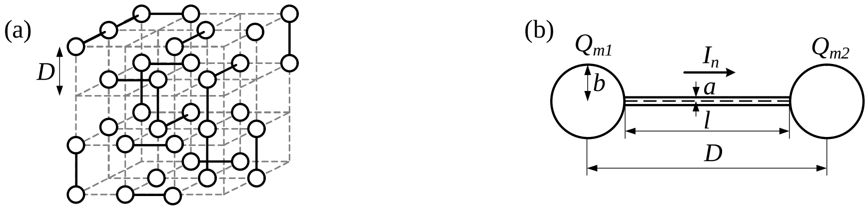

Fig. 1: (a) An example of 3-dimensional structure of conducting spheres connected by wires. Parameter D is the size of the cubic unit cells. (b) Parameter a is the radius and l is the length of the wire. b is the radius of the spheres. Current I and charges Q are on the surface of the wire and spheres, respectively.

G is the Green function given by

$$G(\boldsymbol{r}, t, \boldsymbol{r}', t') \equiv \frac{\delta(t - t' - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{2}$$

Using the equation (1) and the conservation of charge

$$\nabla \cdot \boldsymbol{J} + \frac{\partial}{\partial t}\rho = 0, \tag{3}$$

we derive the equivalent circuit equation.

Now, assuming that $\boldsymbol{J}$ and ρ are uniformly distributed only on the surfaces of the wires and spheres, respectively as Fig.1(b), we represent those as

$$\boldsymbol{J}(\boldsymbol{r}, t) = \sum_{n=1}^{N} I_n(t)\boldsymbol{\Psi}_n(\boldsymbol{r}), \quad \rho(\boldsymbol{r}, t) = \sum_{m=1}^{M} Q_m(t)\Phi_m(\boldsymbol{r}), \tag{4}$$

where $\boldsymbol{\Psi}_n(\boldsymbol{r})$ and $\Phi_m(\boldsymbol{r})$ are normalized basis functions. Replacing the spheres and wires by nodes and edges of the corresponding equivalent circuit, we assume the following relation of normalized basis functions

$$\nabla \cdot \boldsymbol{\Psi}_n = \sum_{m=1}^{M} U_{mn}\Phi_m, \tag{5}$$

where U_{mn} is the incidence matrix which express connections of the circuit. Applying the Galerkin method to the equations (1) and (3), besides neglecting the retardation because of $D \ll \lambda$, we obtain the equivalent circuit equations based on a current vector $\boldsymbol{I}$ and a charge vector $\boldsymbol{Q}$ by

$$\boldsymbol{L}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{I} - \boldsymbol{U}^T\boldsymbol{P}\boldsymbol{Q} = -\boldsymbol{V}^E, \quad \boldsymbol{U}\boldsymbol{I} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q} = \boldsymbol{0}, \tag{6}$$

where $\boldsymbol{L}$ is the inductance matrix, $\boldsymbol{P}$ is the potential matrix which is defined by the capacitance matrix $\boldsymbol{C}$ as $\boldsymbol{P} \equiv \boldsymbol{C}^{-1}$, and $\boldsymbol{V}^E$ is the external source vector. Furthermore, combining the equations (6), we obtain a second order differential equation

$$\boldsymbol{L}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{I} + \boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U}\boldsymbol{I} = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{V}^E. \tag{7}$$

3. Resonant mode and frequency analysis

First, we confirm the method by a simple example shown in Fig.2(a). In order to analyze the resonant modes and frequencies of the structure, we solve the eigenvalue problem of the equation (7). Because $\boldsymbol{L}^{-1}\boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U}$ is a 2×2 matrix, two modes exist, i.e., one is a common mode with its resonant frequency 8.10 GHz and the other is a differential mode with its resonant frequency 8.17 GHz. In order to confirm

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(a) (b)

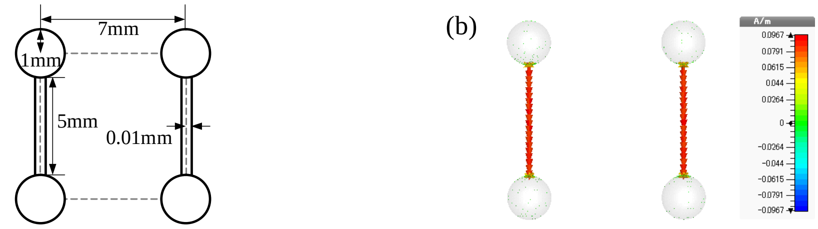

Fig. 2: (a) Four spheres connected by two wires. (b) Current density by the electromagnetic field analysis.

the results, we analyze the current density of the structure using the full-wave electromagnetic field simulation with finite integration technique. Fig.2(b) shows that the current flows only on the surface of the wires and our approximation is valid. The resonant frequencies of each mode are 8.10 GHz and 7.90 GHz, respectively. The errors of each mode between the model and the electromagnetic field analysis are 0.0% and 2.5%.

Second, we confirm the method with a more complicated structure shown in Fig.3(a). The size of conductors are the same in Fig.2(a). The result of current density by the electromagnetic field analysis is shown in Fig.3(b). After enough time has passed, only one mode in Fig.3(a) which is the lowest resonant frequency 3.12 GHz exists in the conductors. The resonant frequency of the mode by our model is 3.08 GHz and the error is 1.3%. Thus, our model gives a good approximation of the resonant frequency.

(a) (b)

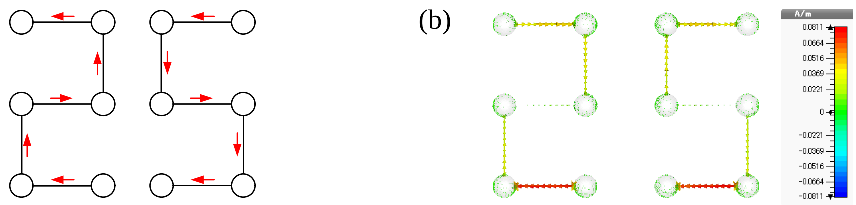

Fig. 3: (a) Twelve spheres connected by ten wires. Arrows show the mode after enough time. (b) Current density by the electromagnetic field analysis.

4. Conclusion

We considered the structure composed of conducting spheres connected by wires, and proposed the systematic method for deriving the equivalent circuit model including the spatial coupling among meta-atoms. Comparing the resonant frequencies by the proposed model with the results of the electromagnetic field analysis, we confirm the validity of the model. The proposed model can be applied to 3-dimensional structures, and enable us to explain various electromagnetic phenomena in metamaterials.

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References

[1] A. Sanada, Characteristics of an isotropic 3-dimensional left-handed metamaterial composed of wired metallic spheres, IEICE Electronics Express, vol. 6, no. 11, pp. 689-702, 2009.

[2] M. E. Verbeek, Partial element equivalent circuit (PEEC) models for on-chip passives and interconnects, *Int. J. Numer. Model.*, vol. 17, pp. 61-84, 2004.

