# Transformation Optics Based Cloaked Sensors

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**Abstract**

Ideal transformation optics cloaking at positive frequency, besides rendering the cloaked region invisible to detection by scattering of incident waves, also shields the region from those same waves. In contrast, we demonstrate that approximate cloaking permits a strong coupling between the cloaked and uncloaked regions; careful choice of parameters allows this coupling to be amplified, leading to effective cloaks with degraded shielding. The sensor modes we describe are close to but distinct from interior resonances, which destroy cloaking. As one application, the sensor mode of a cloak makes it possible to use transformation optics to hide sensors in the cloaked region and yet enable the sensors to efficiently measure waves incident on the exterior of the cloak, an effect similar to the plasmon based approach of Alù and Engheta [3]. We discuss this in 2D, having treated the 3D case in [9].

## 1. Introduction

Transformation optics has led to designs for devices having radical effects on wave propagation, one of the most compelling of which is cloaking [1, 2]. The complex material parameters of a transformation optics cloak steer the rays around the region to be hidden, and to a large extent the behavior of the waves mimics that of the rays. The early studies on such cloaks leaves the impression that transformation optics cloaking produces a decoupling of the waves within and external to the cloaked region, cf. [3, 4, 5, 6, 7, 8]. Later, it has been observed that the cloaked region in fact has a coupling with the environment surrounding the cloak, and this may be amplified by means of carefully chosen parameters within the cloaked region [9]. There is thus a fundamental difference between cloaking for rays and cloaking for waves. We emphasize that the cloaking effectiveness can be made arbitrarily close to the ideal cloak, while keeping the magnitude of the coupling fixed. Although we focus on scalar equations in the quasistatic and finite frequency regimes, modeling electrostatic [10] and acoustic cloaks [11], the same phenomenon holds with regard to transformation optics cloaking for transversely polarized electromagnetic waves, e.g., in the cylindrical geometry [12].

Here, we consider waves modeled by the Helmholtz equation $\nabla\cdot\sigma(x)\nabla u(x)+\kappa(x)\omega^2u(x)=0$ in two and three dimensions. In electrostatics, $\omega=0$ and $\sigma$ denotes the conductivity; in acoustics, $\omega>0$ is the frequency, and $\sigma$ and $\kappa$ correspond to the inverse of the anisotropic mass density and the inverse the of bulk modulus, resp.

At nonzero frequencies, ideal cloaking [2] is accompanied by perfect *shielding*, meaning that an observer or device within the cloaked region cannot detect any information about the incident wave [12]. Furthermore, approximate cloaking is also accompanied by approximate shielding [5, 13] except near certain exceptional frequencies at which the interior of the cloak is resonating. In the 3-dimensional case the resonance frequencies of the approximate cloaks tend to the Neumann eigenvalues of the interior when the approximation becomes better [13]. However, in the 2-dimensional case the resonance frequencies

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of approximate cloaks tend to the eigenvalues corresponding to a non-local boundary condition [16]. In both the 2 and 3 dimensional case both the cloaking and shielding effects are destroyed at the resonance frequencies. In this paper we consider the sensor effect that when the parameters of an object inside the cloak are chosen carefully so that the cloaking effect appear but on the other hand the waves propagate inside the cloaked region and are coupled with the wave outside the cloak, see Fig. 1. We have previously studied the 3D case in [9]; here, we consider the situation in 2D. This transformation optics based effect is similar to the cloaked sensor effect studied by Alù and Engheta for plasmonic cloaks [3].

## 2. Mathematical analysis of the tuned approximative cloaks

*Coordinate transformations* - Let us recall the construction of approximations to ideal cloaks.

Let $B_r = \{\mathbf{x} \in \mathbf{R}^2;\ |\mathbf{x}| < r\}$ denote a disc of radius $r > 0$ and $S_r = \partial B_r$ be its boundary. Let $1 \le R < 2$ be a parameter, and set $\rho = 2(R-1)$, $0 \le \rho < 2$. Then $R \searrow 1$ as $\rho \searrow 0$. Let $F_\rho : B_3 - B_\rho \to B_3 - B_R$ be the coordinate transformation

$$\mathbf{x} := F_\rho(\mathbf{y}) = \begin{cases} \mathbf{y}, & \text{for } 2 < |\mathbf{y}| < 3, \\ \left(1 + \frac{|\mathbf{y}|}{2}\right)\frac{\mathbf{y}}{|\mathbf{y}|}, & \text{for } \rho < |\mathbf{y}| \le 2. \end{cases} \tag{1}$$

For $\rho = 0$, the map $F = F_0$ is the singular transformation of [10, 2], leading to the ideal transformation optics cloak, while for $\rho > 0$, $F_\rho$ is nonsingular and leads to a class of approximate cloaks [5, 13, 15]. Let $\sigma_0 = \delta^{jk}$ denote the homogeneous, isotropic tensor and define for then for $\rho > 0$ the approximate cloak tensor $\sigma_\rho$ on $B_3$ as

$$\sigma_\rho^{jk}(\mathbf{x}) = \begin{cases} \delta^{jk}, & \text{for } |\mathbf{x}| < R \text{ or } 2 < |\mathbf{x}| < 3, \\ \sigma^{jk}(\mathbf{x}), & \text{for } R < |\mathbf{y}| \le 2. \end{cases}$$

Here, $\sigma(x) = (DF(y))\sigma_0(y)\,(DF(y))^t/\det(DF(y))$, $y = F^{-1}(x)$ is the standard cloaking tensor. Define also

$$\kappa_\rho(\mathbf{x}) = \begin{cases} 1, & \text{for } |\mathbf{x}| < R \text{ or } 2 < |\mathbf{x}| < 3, \\ \det(DF_\rho(F_\rho^{-1}(\mathbf{x})))^{1/2}, & \text{for } R < |\mathbf{x}| \le 2. \end{cases}$$

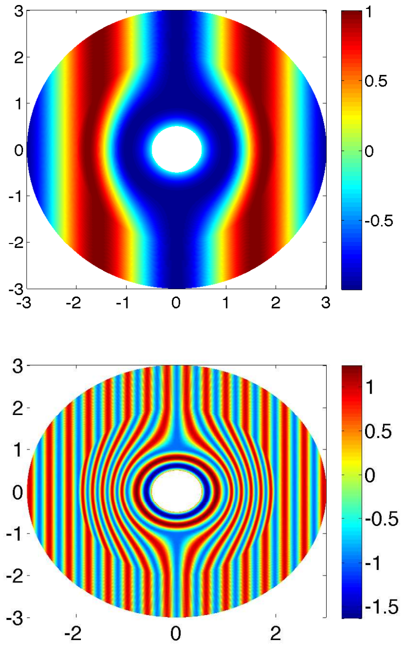


Fig. 1: Plane wave incident to the cloaked sensor for frequency $\omega = 2$ (top) and $\omega = 16$ (bottom) in 3D case.

Now place inside the cloak a scatterer $B_{R_0}$ of radius $R_0 < 1$, with a surface whose impedance induces a real Robin boundary condition on $S_{R_0}$. Thus, we consider in the domain $\Omega = B_3 - B_{R_0}$ the solutions of the boundary value problem,

$$(\nabla\cdot\sigma_\rho\nabla + \omega^2\kappa_\rho)u_\rho = 0 \quad \text{in } \Omega, \quad u_\rho|_{S_3} = f,\ (\partial_r + \alpha)u_\rho|_{S_{R_0}} = 0, \tag{2}$$

for an impedance $\lambda = -i\alpha$ to be specified later. In the domain $2 < |\mathbf{x}| < 3$ we can represent $u_\rho$ in polar coordinates $(r,\theta)$ as $u_\rho(r,\theta) = \sum_{n=-\infty}^{\infty} u_\rho^n(r)e^{in\theta}$. Then equations (2) give rise to a family of boundary value problems for the $u_\rho^n$. For our purposes, the most important one is $u_\rho^0$, i.e., the radial component of $u_\rho$, which is independent of $\theta$, and we study this next.

*Lowest harmonic and the sensor effect* - Consider the problem (2) with $f = const$, so that $u_\rho^\alpha(r,\theta) = u_\rho^\alpha(r)$. The Helmholtz equation (2) leads then to an ordinary differential equation for $u(r) = u_\rho^\alpha(r)$. We can then specify the Cauchy data (i.e., the value of the wave and its normal derivative) at $r = 3$, that is, fix $\left(u(3), \frac{du}{dr}(3)\right) = (f,g)$, solve the function $u(r)$ for $R_0 \le r \le 3$ using the ordinary differential equation and compute the value of the parameter $\alpha$ for this solution by setting $\alpha = -u'(R_0)/u(R_0)$. In other words, for any value $\rho > 0$ and for any pair $(f,g)$ we can determine $\alpha$ so that the Helmholtz equation (2) has a radial solution $u(r)$ with $\left(u(3), \frac{du}{dr}(3)\right) = (f,g)$.

 

In the case when $(f,g) = (0,1)$ we say that the corresponding Robin coefficient $\alpha = \alpha^{res}(\rho)$ gives a value of the impedance which induces a resonant wave, $u^{res}$. Also, in the case when $(f,g) = (J_0(3\omega), \omega J_0'(3\omega)$ we say that the corresponding Robin coefficient $\alpha = \alpha^{sen}(\rho)$ gives a value of the impedance which induces the sensor effect and giving rise to the sensor solution, $u^{sen}$. A calculation shows that $\alpha^{sen}$ is close to $\alpha^{res}$: for some $b_0 \neq 0$, one has $\alpha^{res}(\rho) - \alpha^{sen}(\rho) = b_0\rho + O(\rho^2)$, as $\rho \searrow 0$. Thus, the sensor effect is quite sensitive to perturbations. More careful analysis confirms the following three regimes, depending on how close $\alpha$ is to $\alpha^{sen}(\rho)$:

**(i) Cloaking for generic impedance.** If $\alpha$ is bounded away from $\alpha^{sen}$ i.e., $|\alpha - \alpha^{sen}(\rho)| \geq c > 0$, then the cloak acts as an effective approximate cloak, and the field goes to zero in the cloaked region as $\rho \searrow 0$, so that there is no sensor effect.

**(ii) Resonance effect.** For the specific value $\alpha = \alpha^{res}(\rho)$, the interior resonance leads to both the destruction of cloaking and the absence of shielding, since then $\omega$ is an eigenfrequency of the equation (2) with boundary condition $u = 0$ on $\partial\Omega$ and $(\partial_r + \alpha)u = 0$ on $r = R_0$.

**(iii) Cloaking with sensor effect.** For the value $\alpha = \alpha^{sen}(\rho)$, the cloak acts as an effective approximate cloak, but inside the cloaked region the solution is proportional to the value which the field would have had at the origin in free space, with proportionality of order $O(1)$ as $\rho \to 0$, that is, for $R_0 < r < R$, $u_\rho(r) = c^0(r)v_0 + O(\rho)$, where $c^0(r)$ is a not identically vanishing function and $v_0$ is the value at $O$ of the solution to the free-space $\nabla \cdot \nabla v + \omega^2 v = 0$, $v|_{r=3} = f$. Thus, in the sensor mode the cloak functions as an "invisible magnifying glass", where the value which the field would have had in the empty space at a single point can be measured anywhere inside the cloak. If one places inside the ball $B(R_0)$ a device which measures e.g. the Neumann boundary value of the solution on the boundary $\partial B_{R_0}$, this allows one to enclose a measurement device which does not affect the incident fields being measured, see [9]. We also note that in the sensor mode the cloaking is actually improved as the scattered field has in the region $|\mathbf{x}| > 2$ the magnitude $O(\rho)$ as $\rho \to 0$.

Roughly speaking, when $\alpha = \alpha^{sen}(\rho)$, the frequency $\omega$ is so close to the eigenfrequency of the inside of the cloak that the energy flux from the inside to outside and from the outside to inside through the surface $r = R$ are balanced. The solution inside the cloak does not blow up and the energy flux from the inside cancels the scattering caused by the fact that the cloak is only an approximate cloak, not a perfect cloak.

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