Basics of homogenization of Maxwell equations

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Abstract

Volume or statistical averaging of the microscopic Maxwell equations (MEs), i.e. transition from microscopic MEs to their macroscopic counterparts, is one of the main steps in electrodynamics of materials. In spite of the fundamental importance of the averaging procedure, it is quite rarely properly discussed in university courses and respective books; up to now there is no established consensus about how the averaging procedure has to be performed. In this work we show that there are some basic principles for the averaging procedure (irrespective to what type of material is studied) which have to be satisfied. Any homogenization model has to be consistent with the basic principles. In case of absence of this correlation of a particular model with the basic principles the model could not be accepted as a credible one. Another goal of this work is to establish the averaging procedure for bulk metamaterials, which is rather close to the case of compound materials but should include magnetic response of the inclusions and their clusters.

1. Introduction

Major works on averaging of Maxwell equations (ME) root back to the 50th years of the last century. After the ME for free space had been established, it was realized that their application to real media requires further development. There are two main roads to average of ME: phenomenological [1], [2] and microscopic, and the latter leads to the introducing of multipoles [3]. The problem of homogenization of metamaterials renews interest to the basics of the homogenization of the Maxwell equations. Among a number publications it appears to be unacceptable tradition do not pay attention to the basic principles, which resulted in publication of papers with direct or indirect violation of principles of causality and/or passivity [4]. The problem is amplified by the fact that the theory of homogenization of ME is extremely poorly presented in university courses even for the established cases of compound materials in case of static approximation (Maxwell-Garnett-Lorenz-Lorentz and Brügermann models).
From the other side, the homogenization model for ME can be developed (to some extend) irrespectively of a particular type of media, which in turn means that any particular model has to be consistent with these general rules. In our work we propose a methodology of the homogenization procedure for ME, which results in three possible forms of macroscopic (averaged, homogenized) ME. We have found, that in addition to the known Casimir and Landau&Lifshitz representations, the ME possess another form, which we named "Anapole" representation. We argue that any homogenization model has to finally result in one of these three forms. We show that these macroscopic equations are invariant with respect to a particular type of transformation, which can be reduced to the known Serdyukov-Fedorov transformation. We also show that the transformation allows us in some cases to transit from one to another form of representations, and we have established general rules for these transformations. We show how phenomenological and microscopic homogenization approaches in general relate to each other and how the microscopic (multipole-based) approach can be extended to the case of homogenization of nonlinear and active metamaterials.

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2. Possible representation of the Maxwell equations

We consider propagation of an electromagnetic wave interacting with the medium in case when the classical dynamics is supposed to be valid and the bulk material fills the whole space; the ME in this case can be formally averaged over a physically small volume (or through statistic averaging), which results in:

$$
\begin{cases}
\operatorname{rot}\langle\vec{e}\rangle = \frac{i\omega}{c}\langle\vec{h}\rangle \\
\operatorname{div}\langle\vec{h}\rangle = 0 \\
\operatorname{div}\langle\vec{e}\rangle = 4\pi\langle\rho\rangle \\
\operatorname{rot}\langle\vec{h}\rangle = -\frac{i\omega}{c}\langle\vec{e}\rangle + \frac{4\pi}{c}\langle\vec{\mathrm{j}}\rangle \\
\rho = \sum_i q_i \delta\left(\vec{r}-\vec{r}_i\right) \\
\langle\vec{\mathrm{j}}\rangle = \sum_i q_i \left\langle \vec{v}_i \delta\left(\vec{r}-\vec{r}_i\right)\right\rangle \\
\frac{d\vec{v}_i}{dt} = \frac{q_i}{m_i}\vec{e} + \frac{q_i}{m_i c}\left[\vec{v}_i * \vec{h}\right]
\end{cases}
\quad
\xrightarrow[\langle\vec{h}\rangle = \vec{B}]{\langle\vec{e}\rangle = \vec{E}}
\quad
\begin{cases}
\operatorname{rot}\vec{E} = \frac{i\omega}{c}\vec{B} \\
\operatorname{div}\vec{B} = 0 \\
\operatorname{div}\vec{E} = 4\pi\langle\rho\rangle \\
\operatorname{rot}\vec{B} = -\frac{i\omega}{c}\vec{E} + \frac{4\pi}{c}\langle\vec{\mathrm{j}}\rangle \\
\langle\rho\rangle = \sum_i q_i \delta\left(\vec{r}-\vec{r}_i\right) = \langle\rho\rangle\left(\vec{E},\vec{B}\right) \\
\langle\vec{\mathrm{j}}\rangle = \sum_i q_i \left\langle \vec{v}_i \delta\left(\vec{r}-\vec{r}_i\right)\right\rangle = \langle\vec{\mathrm{j}}\rangle\left(\vec{E},\vec{B}\right) \\
\frac{d\vec{v}_i}{dt} = \frac{q_i}{m_i}\vec{e} + \frac{q_i}{m_i c}\left[\vec{v}_i * \vec{h}\right]
\end{cases}
\tag{1}
$$

The main problem here is to find the averaged current and charge distribution as functions of the averaged electric and magnetic fields:

$$
\begin{gathered}
\langle\vec{\mathrm{j}}\rangle = \langle\vec{\mathrm{j}}\rangle\left(\vec{E},\vec{B}\right) \\
\langle\rho\rangle = \langle\rho\rangle\left(\vec{E},\vec{B}\right)
\end{gathered}
\tag{2}
$$

Casimir form ("C" form) $\vec{E},\ \vec{B},\ \vec{D},\ \vec{H}$	**Landau & Lifshitz form ("LL" form)** $\vec{E},\ \vec{B},\ \vec{D},$ ~~$\vec{H}$~~	**Anapole form ("A" form)** $\vec{E},\ \vec{B},$ ~~$\vec{D}$~~, $\vec{H}$
$\begin{cases}\langle\rho\rangle = -\operatorname{div}\vec{P}_C \\ \langle\vec{j}\rangle = -i\omega\vec{P}_C + c\operatorname{rot}\vec{M}_C\end{cases}$ $\begin{cases}\vec{D} = \vec{E} + 4\pi\vec{P}_C \\ \vec{H} = \vec{B} - 4\pi\vec{M}_C\end{cases}$ $\begin{cases}\operatorname{rot}\vec{E} = \frac{i\omega}{c}\vec{B} \\ \operatorname{div}\vec{B} = 0 \\ \operatorname{div}\vec{D} = 0 \\ \operatorname{rot}\vec{H} = -\frac{i\omega}{c}\vec{D}\end{cases}$	$\begin{cases}\langle\rho\rangle = -\operatorname{div}\vec{P}_{LL} \\ \langle\vec{j}\rangle = -i\omega\vec{P}_{LL}\end{cases}$ $\begin{cases}\vec{D} = \vec{E} + 4\pi\vec{P}_{LL} \\ \vec{B} = \vec{B}\end{cases}$ $\begin{cases}\operatorname{rot}\vec{E} = \frac{i\omega}{c}\vec{B} \\ \operatorname{div}\vec{B} = 0 \\ \operatorname{div}\vec{D} = 0 \\ \operatorname{rot}\vec{B} = -\frac{i\omega}{c}\vec{D}\end{cases}$	$\begin{cases}\langle\rho\rangle = 0 \\ \langle\vec{j}\rangle = c\operatorname{rot}\vec{M}_A\end{cases}$ $\begin{cases}\vec{D} = \vec{E} \\ \vec{H} = \vec{B} - 4\pi\vec{M}_A\end{cases}$ $\begin{cases}\operatorname{rot}\vec{E} = \frac{i\omega}{c}\vec{B} \\ \operatorname{div}\vec{B} = 0 \\ \operatorname{div}\vec{E} = 0 \\ \operatorname{rot}\vec{H} = -\frac{i\omega}{c}\vec{E}\end{cases}$

3. Transformations between representations

The transformations below can be considered as a natural consequence of the present approach and provides transformation between two realizable physical situations ($\vec{T}_1 \neq 0, \vec{T}_2 = 0$ are arbitrary differentiable vectors) or between two representations of the same physical situation ($\vec{T}_1 = 0, \vec{T}_2 \neq 0$):

$$\begin{cases} \vec{B} = \vec{B}' + \mathrm{rot}\,\vec{T}_1 \\ \vec{E} = \vec{E}' + \dfrac{i\omega}{c}\vec{T}_1 \end{cases} \quad \begin{cases} \vec{P} = \vec{P}' - \dfrac{i\omega}{4\pi c}\vec{T}_1 + \dfrac{\mathrm{rot}\,\vec{T}_2}{4\pi} \\ \vec{M} = \vec{M}' + \dfrac{1}{4\pi}\mathrm{rot}\,\vec{T}_1 - \dfrac{i\omega}{c}\vec{T}_2 \end{cases} \quad \begin{cases} \langle\rho\rangle = \langle\rho\rangle' + \dfrac{i\omega}{4\pi c}\mathrm{div}\,\vec{T}_1 \\ \langle\vec{j}\rangle = \langle\vec{j}\rangle' + \dfrac{c}{4\pi}\mathrm{rot}\,\mathrm{rot}\,\vec{T}_1 - \dfrac{\omega^2}{4\pi c}\vec{T}_1 \end{cases} \tag{3}$$

Using transformations (3) one can show that the presented in Table 1 representations are mutually transformable, but these transformations are not arbitrary and follow the rules shown below:

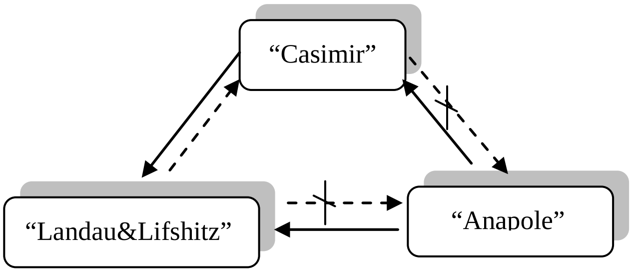

Fig. 1: Possibility of mutual transformation between different representations. Crossed dashed lines between "Casimir" and "Anapole" and "Landau&Lifshitz" and "Anapole" show impossible transformations, multiple dashed lines between "L&L" and "C" show not unique transformations.

One can show that within the frame of the phenomenological approach and weak spatial dispersion, the Casimir form can be reduced to the following set of material equations:

$$\begin{cases} D_{C,\alpha}\left(\vec{k},\omega\right) = \left(\varepsilon^{(0)}_{C,\alpha\beta}(\omega) + \varepsilon^{(1)}_{C,\alpha\beta\gamma}(\omega)k_\gamma + \varepsilon^{(2)}_{C,\alpha\beta\gamma\delta}(\omega)k_\gamma k_\delta\right)E_\beta\left(\vec{k},\omega\right) \\ H_\alpha\left(\vec{k},\omega\right) = \left(\phi^{(0)}_{C,\alpha\beta}(\omega) + \phi^{(1)}_{C,\alpha\beta\gamma}(\omega)k_\gamma\right)E_\beta\left(\vec{k},\omega\right) \end{cases} \tag{4}$$

which under some further assumptions can be in turn reduced to the well-known form of Post (e.g., [5]) material equations:

$$\begin{cases} D_{C,\alpha}\left(\vec{k},\omega\right) = \left(\varepsilon^{(0)}_{C,\alpha\beta}(\omega) + \varepsilon^{(1,sym)}_{C,\alpha\beta\gamma}(\omega)k_\gamma\right)E_\beta\left(\vec{k},\omega\right) + G^{(\varepsilon)}_{C,\alpha\beta}(\omega)B_\beta\left(\vec{k},\omega\right) \\ H_\alpha\left(\vec{k},\omega\right) = G^{(\phi)}_{C,\alpha\beta}(\omega)B_\beta\left(\vec{k},\omega\right) + \phi^{(0)}_{C,\alpha\beta}(\omega)E_\beta\left(\vec{k},\omega\right) \end{cases} \tag{5}$$

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