Transformation thermodynamics: playing with fire

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Abstract

It has been recently proposed to control heat fluxes using tools of transformation optics in the context of thermodynamics [1]. A thermic cloak protecting a region from heat and a concentrator increasing gradient of temperature in a region have been theorized. Here, we propose an extension of this work to time-harmonic heat sources.

1. Introduction

There is currently a keen interest in transformational optics following the work of two research groups (those of Pendry, Schurig and Smith [2] and Leonhardt [3]), which independently proposed a systematic way to control light trajectories in curvilinear coordinate systems. In order to attract attention of mass media, these groups designed a cloak that renders any object inside it invisible to electromagnetic radiation. The former team theoretized that a coating consisting of a meta-material whose physical properties are deduced from a coordinate transformation in the Maxwell system displays anisotropy and heterogeneity of permittivity and permeability working as a deformation of the optical space around the object. The physicists consider the blowup of a point, thereby tearing apart the metric space. Though this may seem haphazardous, this can be legitimated by making use of advanced mathematical treatments [4, 5].

2. Transformed time-harmonic heat equation

We consider the time-harmonic diffusion equation in a bounded cylindrical domain Ω

$$\imath\omega u = \nabla \cdot (\kappa \nabla u) \, , \frac{\partial u}{\partial n} \mid \partial\Omega = g \tag{1}$$

where κ is the thermal diffusivity and with a source outside corresponding to an input heat flux $g(x, y)e^{\omega t}$ on the boundary $\partial\Omega$ of Ω, with t the time variable and ω the angular frequency. It is customary to let κ go in front of the spatial derivatives when the medium is homogeneous. However, here we consider a heterogeneous medium, hence the spatial derivatives of κ might suffer some discontinuity (derivatives are taken in distributional sense, hence transmission conditions ensuring continuity of the heat flux $\kappa\nabla u$ are encompassed in (1)).

Upon a change of variable $(x, y) \to (x', y')$ described by a Jacobian matrix $\mathbf{J} = \partial(x', y')/\partial(x, y)$, this equation takes the form:

$$\det(\mathbf{J})\imath\omega u = \nabla \cdot \left(\mathbf{J}^{-T} \kappa \mathbf{J}^{-1} \det(\mathbf{J}) \nabla u \right) \; . \tag{2}$$

We note that (1) and (2) have the same structure, except that the transformed diffusivity

$$\underline{\underline{\kappa'}} = \mathbf{J}^{-T} \kappa \mathbf{J}^{-1} \det(\mathbf{J}) = \kappa \mathbf{J}^{-T} \mathbf{J}^{-1} \det(\mathbf{J}) = \kappa \mathbf{T}^{-1} \; , \tag{3}$$

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is matrix-valued, with $\mathbf{T}$ the metric tensor, and the time derivative in the left hand side is multiplied by the determinant of the Jacobian matrix $\mathbf{J}$ of the transformation. We note that this equation was not studied previously, albeit some similarities with the time-harmonic Schrödinger equation [6].

3. Illsutrative numerical examples

Let us now consider two paradigms of transformation optics, the cloak and the concentrator [7], in the context of thermodynamics.

Following the work by Pendry et al. [2], we consider the linear geometric transform:

$$\begin{cases} r' &= \frac{R_2-R_1}{R_2} r + R_1 \\ \theta' &= \theta \end{cases} \tag{4}$$

which is simply a radial stretch of polar coordinates mapping a disc of radius $r = R_2$ onto a coronna $R_1 \leq r' \leq R_2$.

The transformed diffusivity inside the circular coating of the cloak can be expressed as

$$\underline{\underline{\kappa}}' = \kappa \mathbf{T}^{-1} = \mathbf{R}(\theta')\mathrm{diag}(\kappa'_{r'}, \kappa'_{\theta'})\mathbf{R}(\theta')^T \tag{5}$$

with $\mathbf{R}(\theta)$ the rotation matrix through an angle θ and where the eigenvalues of the diagonal matrix (principal values of diffusivity) are

$$\kappa'_{r'} \quad = \frac{r' - R_1}{r'} \;, \quad \kappa'_{\theta} = \frac{r'}{r' - R_1} \;. \tag{6}$$

We show in Fig. 1 (left panel) the result of some simulation with COMSOL MULTIPHYSICS. The cloak indeed protects a region from heat.

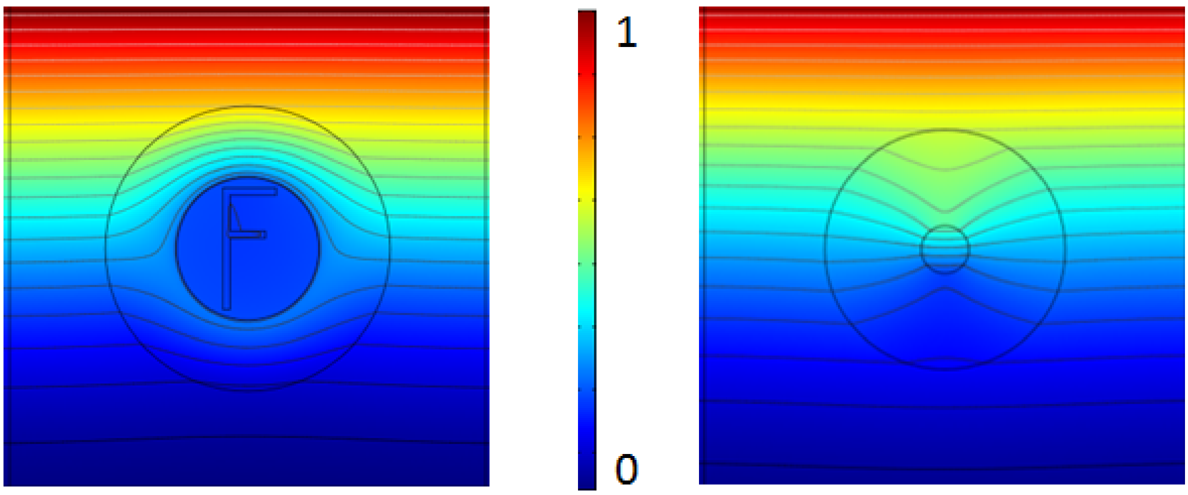

Fig. 1: Left: A thermic cloak in presence of a time-harmonic heat source on top (with temperature is normalized to 1) protects a conducting F-shaped object located in its core from heat fluxes; Right: A thermic concentrator increases heat fluxes in its core.

Let us now design a concentrator for heat. Here, we consider three embedded discs defined by $R_1 < R_2 < R_3$. The transformation now maps the field in the region $0 \leq r \leq R_2$ onto $0 \leq r' \leq R_1$

(i.e. compression of thermal space) and the field in the region $R_2 \leq r \leq R_3$ onto $R_1 \leq r' \leq R_3$ (i.e. extension of thermal space). Importantly, the compression and extension compensate each other for $0 < r \leq R_3$ and the transformation should be the identity for $R_3 < r$. We choose the transformation proposed in [7].

This leads us to $\mathbf{T}^{-1} = \mathbf{R}(\theta)\mathbf{Diag}(\kappa'_r, \kappa'_\theta)\mathbf{R}(\theta)^T$ where the cloak is described by the following parameters

$$\begin{aligned} \kappa'_r &= 1\ , & \kappa'_\theta &= 1\ , & &\text{if } 0 \leq r' \leq R_2 \\ \kappa'_r &= \frac{r' + R_3 \frac{R_2 - R_1}{R_3 - R_2}}{r'}\ , & \kappa'_\theta &= \frac{r'}{r' + R_3 \frac{R_2 - R_1}{R_3 - R_2}}\ , & &\text{if } R_2 \leq r' \leq R_3 \end{aligned} \tag{7}$$

where R_1 and R_2 are the interior and the exterior radii of the cloak.

We show in Fig. 1 (right) the result of the simulation in COMSOL MULTIPHYSICS in that case. One can clearly see that the gradient of temperature is enhanced in the inner core of the cocentrator.

4. Conclusion

We have successfully applied concepts of transformation optics to the area of thermodynamics for the paradigms of a thermic cloak and concentrator. We focussed in on time-harmonic heat sources, while we studied earlier the case of transient heat sources. The next step is to validate experimentally our theoretical proposals.

References

[1] S. Guenneau, C. Amra and D. Veynante, Transformation thermodynamics: cloaking and concentrating heat flux, *Optics Express,* vol. 20, p. 8207, 2012.
[2] J.B. Pendry, D. Schurig, and D.R. Smith, Controlling Electromagnetic Fields, *Science,* vol. 312, p. 1780, 2006.
[3] U. Leonhardt, Optical Conformal Mapping, *Science,* vol. 312, p. 1777, 2006.
[4] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Full-wave invisibility of active devices at all frequencies, *Communications Mathematical Physics,* vol. 275, p. 749, 2007.
[5] R.V. Kohn, H. Shen, M.S. Vogelius and M.I. Weinstein, Cloaking via change of variables in electric impedance tomography, *Inverse Problems,* vol. 24, p. 015016, 2008.
[6] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Isotropic transformation optics: approximate acoustic and quantum cloaking, *New J. Phys.* vol. 10, p. 115024, 2008.
[7] M. Rahm, D. Schurig, D. A. Roberts, S. A. Cummer, D. R. Smith, and J. B. Pendry, Design of electromagnetic cloaks and concentrators using form-invariant coordinate transformations of Maxwells equations, *Photonics Nanostructures Fundamental Applications,* vol. 6, p. 87, 2008.

