Purcell effect in hyperbolic media

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Abstract

We study theoretically the properties of hyperbolic medium modelled as cubic lattices of uniaxial resonant dipoles. In this regime the structure is characterized by hyperbolic isofrequency surfaces in wavevector space, providing huge enhancement of spontaneous emission rate. We investigate the dependence of the Purcell factor on the source position in the lattice unit cell and visualize the collective origin of the Purcell factor enhancement by the lattice Green function.

1. Introduction

Engineering of light-matter coupling in nanostructured environment is nowadays in the focus of active studies [1, 2]. Recently the rapt attention of researchers has been attracted to the so-called hyperbolic medium: uniaxial medium, where the main components of dielectric tensor have different sign [3]. Resulting hyperbolic surface of the wavevectors at given frequency has infinite area, which means infinite photonic density of states and seemingly diverging spontaneous emission rate [2, 3]. Existing experimental reports on the Purcell factor enhancement in hyperbolic metamaterials [2] as well as the theoretical studies for various models [4, 5, 6] indicate, that the Purcell factor remains finite due to the presence of some cutoff in the wavevector space, steming either from spatial inhomogeneity of the medium, or from the finite distance from the light source to the medium.

A number of important questions still remain open, including the structure of the field radiated by the source and the effect of the source position in the unit cell of actual discrete metamaterial on the Purcell factor and Lamb shift. The goal of this work is to answer these questions.

2. Model

We consider the simplest possible theoretical model of the hyperbolic medium: the infinite cubic crystal of resonant point dipoles, polarizable only in z direction. For negative polarizability this model yields hyperbolic isofrequency surfaces and, despite certain idealization, keeps the required physics intact. It has been successfully applied to the lattices of split-ring resonators [7]. External dipole $\boldsymbol{p}_0$ is placed at the point $\boldsymbol{r}_0$, see Fig. 1a. Electric field in the structure satisfies the following equation

$$\nabla \times \nabla \times \boldsymbol{E} - q^2 \boldsymbol{E} = 4\pi q^2 \boldsymbol{P} \,, \quad \boldsymbol{P} = \boldsymbol{d}_0 \delta(\boldsymbol{r} - \boldsymbol{r}_0) + \sum_j \delta(\boldsymbol{r} - \boldsymbol{r}_j) \boldsymbol{p}_j \,. \tag{1}$$

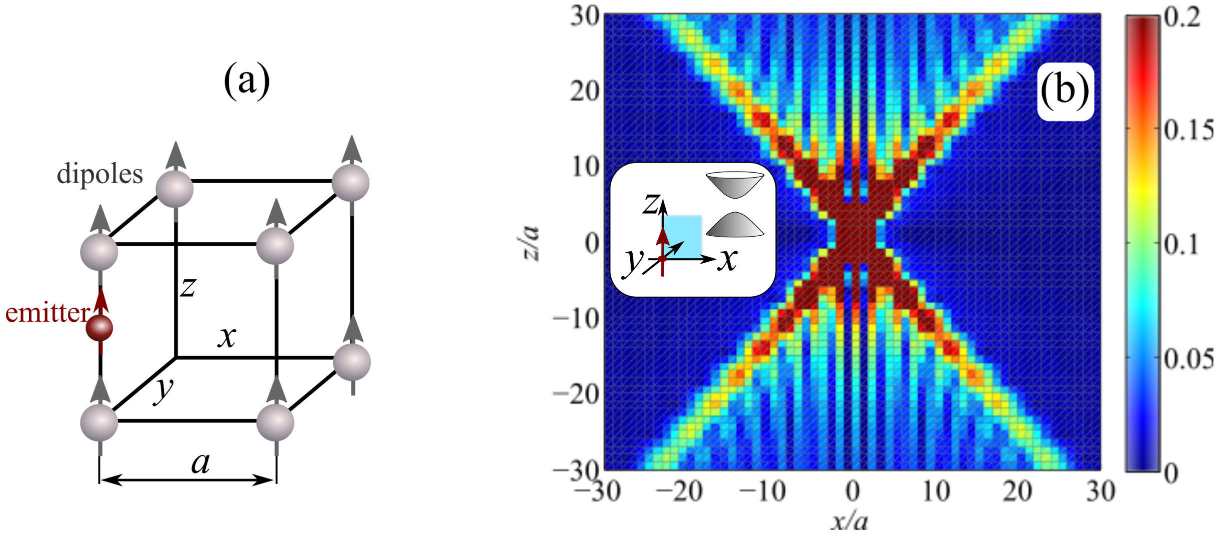

Fig. 1: (a) Schematic illustration of the unit cell of the cubic dipole lattice with period a, where a light source is embedded. (b) Spatial distribution of the dipole moments $|p_z(\boldsymbol{r})|$ in the hyperbolic medium, excited by the point emitter in the plane $y = 0$. Insets schematically illustrate the geometry and the isofrequency surfaces in wavevector space. Calculated for the emitter at $\boldsymbol{r}_0 = 0.5a\boldsymbol{e}_z$ at $qa = 0.15\pi$ and $4\pi\alpha_{0,zz} = -6a^3$, where $\alpha_{0,zz}^{-1} = \alpha_{zz}^{-1} + 2\mathrm{i}q^3/3$.

where $q = \omega/c$ is the wavevector at the frequency ω and $\boldsymbol{P}$ is the net polarization of the lattice dipoles and the probe one. All the dipoles $\boldsymbol{p}_j$ are characterized by the same polarizability $\boldsymbol{p}_j = \boldsymbol{e}_z\alpha_{zz}\boldsymbol{E}_{\text{ext}}(\boldsymbol{r}_j)$, describing their responce to the external electric field.

The total electric field and polarizations, induced in the structure by the probe dipole $\boldsymbol{p}_0$ can be straightforwardly determined by expanding the fields over the Bloch eigenmodes.

3. Results

Spatial distribution of the dipole moments $|p(\boldsymbol{r}_j)|$, excited in the $y = 0$ plane, is shown in Fig. 1b. The Green function pattern has a distinct cross-like shape, typical for hyperbolic medium [3]. The intensity is concentrated within the cone $(x^2 + y^2)\varepsilon_{zz} + z^2 < 0$, where $\varepsilon_{zz} = -1$ is the effective dielectric constant. Strong spatial modulation of the pattern, manifested as distinct vertical stripes in Fig. 1b, reflects the discrete problem geometry and is due to the interference of the Bloch waves with $k_x = \pm\pi/a$ corresponding to the boundary of the Brillouin zone.

Fig. 2 presents the comparison of the Purcell factor calculated for elliptic (a) and hyperbolic (b) regimes as function of the source position z on the vertical edge of the unit cell. Our theory indicates that the Purcell factor is determined by (i) the strength of the near field of the nearest neighbors of the emitter and (ii) the density of photonic states. In both elliptic and hyperbolic cases the Purcell factor diverges at the corners as $f \propto 1/z^6$, proportional to the square of the local field of the nearest neighbor. Similar dependence has been obtained for the Lamb shift. In elliptic medium the Purcell factor is almost completely determined by nearest neighbor, cf. solid and dashed curves in Fig. 2a. However, in hyperbolic medium a large number of neighbors are efficiently excited (see Fig. 1b), or, equivalently, the density of photonic states is high. As a result, the Purcell factor is much larger than the single-neighbor contribution, see Fig. 2b. Enhancement of Purcell factor as compared to elliptic case was estimated for $qa \ll 1$ as $f_{\text{hyp}}/f_{\text{ell}} = \lambda^3/(16a^3)$ where $\lambda = 2\pi c/\omega$, which agrees with the results obtained for distributed source in effective medium approximation [4].

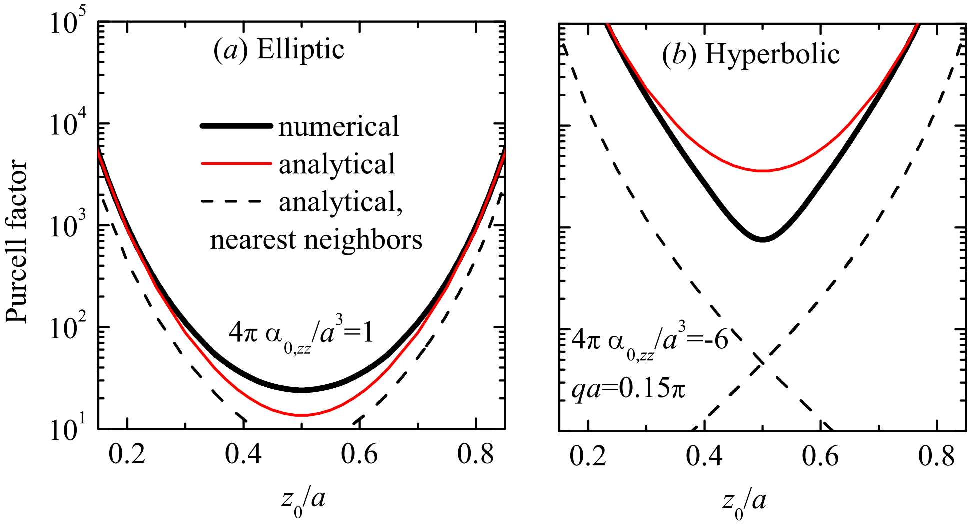

Fig. 2: Purcell factor in (a) elliptic and (b) hyperbolic media as a function of the source coordinate z_0 for $x_0 = y_0 = 0$. Thick solid black, thin solid red and dashed black curves correspond to numerical calculation, analytical theory, and nearest-neighbor approximation, respectively. Two dashed curves correspond to the nearest neighbors at $x = y = z = 0$ and $x = y = 0$, $z = a$. Other parameters are indicated on graph.

4. Conclusion

We have developed the analytical theory of light-matter coupling in discrete hyperbolic medium in the framework of the model of a cubic lattice of uniaxial resonant dipoles. We have calculated Purcell factor, Lamb shift, and Green function, and also revealed that the optimal emitter position is in the local field maxima, close to the lattice nodes.

The density of states is drastically enhanced in the hyperbolic regime as compared to vacuum or single resonant dipole case. As a result, a huge number of lattice dipoles are efficiently excited by the emitter, which has been visualized by calculating the Green function of the lattice. The Green function has a shape of a conus: the field propagates along the directions close to symmetry axis z and decays in the xy plane. Discrete character of the problem results in the strongly spatial modulation of the Green function.

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