Transmission Line Model of Noise: Application to Negative-index Metamaterials

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Abstract

A direct transmission line model of noise in 1D electromagnetic media is presented, and used to find the emittance and noise factor of a negative index metamaterial based on resonators and rods. The results highlight the poor noise performance of negative index materials based on low-Q magnetic resonators.

1. Introduction

Since the time of Johnson [1] and Nyquist [2] it has been known that dissipative electrical elements give rise to thermal noise, due to an inescapable linkage: the fluctuation dissipation theorem [3]. Rytov [4] has shown that the noise propagates as electromagnetic (EM) waves in lossy media, and that this thermal radiation is responsible for the emittance of a body. The essence of his approach was to add randomly varying source terms to the Maxwell curl equations. The sources excite waves (essentially Green's functions), and Rytov found the thermal radiation by summing the waves, taking into account their lack of correlation. Although very general, his method has the limitation that the Green's functions are only known analytically for a few systems. Furthermore, he carried out the summations in k-space rather than real space. Calculations for finite or inhomogeneous media are therefore difficult. Recourse is therefore often made to an indirect method based on Kirchhoff's radiation law, replacing the problem of finding the emittance with that of finding the absorbance. This method again has limitations; it cannot find the noise inside a body, or separate it into electrically and magnetically generated components.

Because most metamaterials are lossy, the question of their thermal performance is of fundamental importance for applications. For example, we would expect the signal-to-noise ratio of a wave passing through a metamaterial slab (Fig. 1) to deteriorate, due to the additive effect of noise. We would also expect to be able to control the spectral variation of emittance to some extent, for example to alter the radiation in a given band. However, effective methods of calculating noise are needed. Some direct methods do exist [5]. Here we demonstrate a simple model for 1D systems, capable of performing full direct and indirect calculations, and demonstrate its use with a negative index slab.

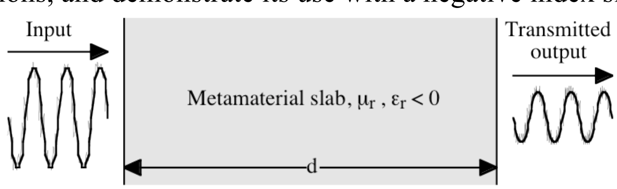

Fig. 1. EM wave incident on a metamaterial slab.

2. 1D Transmission line model

Transmission-line models of EM waves propagating in inhomogeneous media are well known [6]. To apply this approach to noise, 1D space is first discretised into sections of length a. Generally there will be N sections; Fig. 2 shows an example with five. The series impedance in the n^{th} element is written as an inductance of value $L_n = \mu_n a$, where $\mu_n = \mu_n' - j\mu_n''$ is the local permeability. The element has an associated voltage source V_{Mn}, describing magnetic noise, whose RMS value in a frequency interval df is defined at low frequency from the relation $V_{Mn}V_{Mn}^* = 4KT\omega\mu_n''a\ df$, where K is Boltzmann's constant and T is absolute temperature [2]. Similarly, the shunt impedance is a capacitance of value $C_n = \varepsilon_n a$, where $\varepsilon_n = \varepsilon_n' - j\varepsilon_n''$ is the local permittivity. The element has an associated current source I_{En}, describing electric noise, whose RMS value is specified by $I_{En}I_{En}^* = 4KT\omega\varepsilon_n''a\ df$. The local refractive index and impedance are $n = \sqrt{(\mu_n\varepsilon_n)}$ and $Z_n = \sqrt{(\mu_n/\varepsilon_n)}$. The line is terminated at each end with the free space impedance $Z_0 = \sqrt{(\varepsilon_0/\mu_0)}$, which has an associated noise source I_{FS} whose RMS value is specified by $I_{FS}I_{FS}^* = (4KT/Z_0)\ df$.

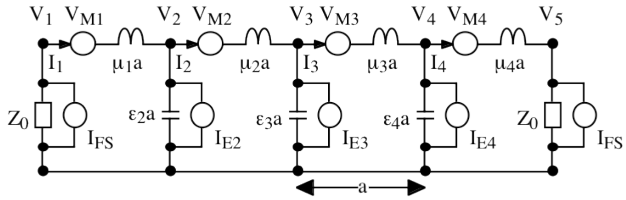

Fig. 2. Transmission line model of a 1D noisy electromagnetic medium.

Within the line, the transmission line equations at angular frequency ω are:

$$V_{n+1} - V_n = -I_n j\omega\mu_n a + V_{Mn}\ ;\ I_n - I_{n-1} = -V_n j\omega\varepsilon_n a + I_{En} \tag{1}$$

These equations are clearly analogous to the Maxwell curl equations, with V_n and I_n representing electric and magnetic fields, respectively, and V_{Mn} and I_{En} representing Rytov's sources. However, for the first and last elements, we have $I_1 = -V_1/Z_0 + I_{FS}$; $-I_{N-1} = -V_N/Z_0 + I_{FS}$. The equations can be written in matrix form as $\underline{M}\ \underline{Y} = \underline{X}$. Here $\underline{M}$ is a N x N matrix, $\underline{Y}$ is a N-element column vector containing the nodal voltages and line currents, and $\underline{X}$ is a N-element column vector containing the noise voltages and currents. Equation 2 shows the matrix representation of Figure 2.

$$\begin{bmatrix} 1/Z_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & j\omega\mu_1 a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & j\omega\varepsilon_2 a & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & j\omega\mu_2 a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & j\omega\varepsilon_3 a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & j\omega\mu_3 a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & j\omega\varepsilon_4 a & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & j\omega\mu_4 a & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1/Z_0 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \\ V_2 \\ I_2 \\ V_3 \\ I_3 \\ V_4 \\ I_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} I_{FS} \\ V_{M1} \\ I_{E2} \\ V_{M2} \\ I_{E3} \\ V_{M3} \\ I_{E4} \\ V_{M4} \\ I_{FS} \end{bmatrix} \tag{2}$$

The unknowns can clearly be found for any input as $\underline{Y} = \underline{M}^{-1}\ \underline{X}$. The local noise due to the material may therefore be found directly, by evaluating $\underline{Y}$ for each source in turn and incoherently summing the results. If the current in element n due to a magnetic noise source V_{Mm} is I_{nMm}, and the current due to an electric noise source I_{Em} is I_{nEm}, the local noise power is $P_n = P_{nE} + P_{nM}$, where $P_{nE} = {}_m\Sigma\ I_{nEm}I_{nEm}^*\ \mathrm{Re}(Z_n)$ and $P_{nM} = {}_m\Sigma\ I_{nMm}I_{nMm}^*\ \mathrm{Re}(Z_n)$, and the sums are taken over all source positions. Calculation of the power P_N dissipated in the right-hand load allows direct calculation of the emittance $E = P_N/KTdf$. Other quantities may be found by evaluating $\underline{Y}$ when the noise arises instead from (say) the left-hand source. In this case, the reflection coefficient is $r = (V_1/I_1 - Z_0)/(V_1/I_1 + Z_0)$ and the transmission coefficient is $t = (1 + r)V_N/V_1$. The reflectance and transmittance are $R = |r|^2$ and $T = |t|^2$, and the noise factor is $F = 1 + E/T$.

3. Example: Negative index slab

To demonstrate the method, we consider a uniform slab of thickness d = Na of a negative-index medium consisting of uncoupled split-ring resonators and rods. In this case the relative permittivity $\varepsilon_{rn} = \varepsilon_n/\varepsilon_0$ and relative permeability $\mu_{rn} = \mu_n/\mu_0$ may be taken as having the constant values:

$$\varepsilon_{rn} = 1 - \omega_p^2/(\omega^2 - j\omega\omega_c)\ ;\ \mu_{rn} = 1 - q^2/(1 - \omega_0^2/\omega^2 - j\omega_0/\omega Q_0) \tag{3}$$

Here ω_0 is the resonant frequency of the resonators, Q_0 is their quality factor and q^2 the filling factor, and ω_p and ω_c are the plasma frequency and collision damping frequency of the rods.

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Since the slab is uniform, we may also compare the results with the prediction of standard EM theory. The transmission and reflection coefficients of a Fabry-Perot cavity are:

$$t = t_{12}t_{21} \exp(-jk_0nd)/\{1 - r_{21}^2 \exp(-j2k_0nd)\} \ ; \ r = r_{12} + t_{12}t_{21}r_{21} \exp(-j2k_0nd)/\{1 - r_{21}^2 \exp(-j2k_0nd)\} \quad (4)$$

Here $k_0 = \omega/\sqrt{(\varepsilon_0\mu_0)}$ is the propagation constant of free-space and r_{ij} and t_{ij} are the transmission and reflection coefficients at the interface between media i and j, given by:

$$r_{12} = (1 - n/\mu_r) / (1 + n/\mu_r) = -r_{21} \ ; \ t_{12} = 2 / (1 + n/\mu_r); \ t_{21} = 2(n/\mu_r) / (1 + (n/\mu_r)) \quad (5)$$

The transmittance and reflectance are again $R = |r|^2$ and $T = |t|^2$, while the absorbance is $A = 1 - T - R$. Invoking Kirchhoff's law, the noise factor can be found this time as $F = 1 + A/T$.

Results are presented for the typical parameters of a = 1 cm, N = 100 (so that d = 1 m), $\omega_0 = c/10a$ (corresponding f_0 = 480 MHz), $Q_0 = 200$, $q^2 = 0.1$, $\omega_p/\omega_0 = 1.1$ and $\omega_\tau/\omega_0 = 0.01$. The parameters are chosen so that ε_r and μ_r are both negative just above ω_0, and in this region the real part of n is also negative. Fig. 3 shows the frequency variations of a) ε_r, b) μ_r, c) n, d) the magnetic, electric and total contributions to the emittance, e) the transmittance, reflectance and absorbance and f) the noise figure NF = 10 $\log_{10}(F)$. Fig. 3d is calculated using the transmission line model, and Figs. 3e and 3f using both models (with identical results). The results show that noise is emitted in the negative index band, from both electric and magnetic sources but mainly the latter. In this range, the noise figure is significant, and this aspect will require careful investigation for practical applications.

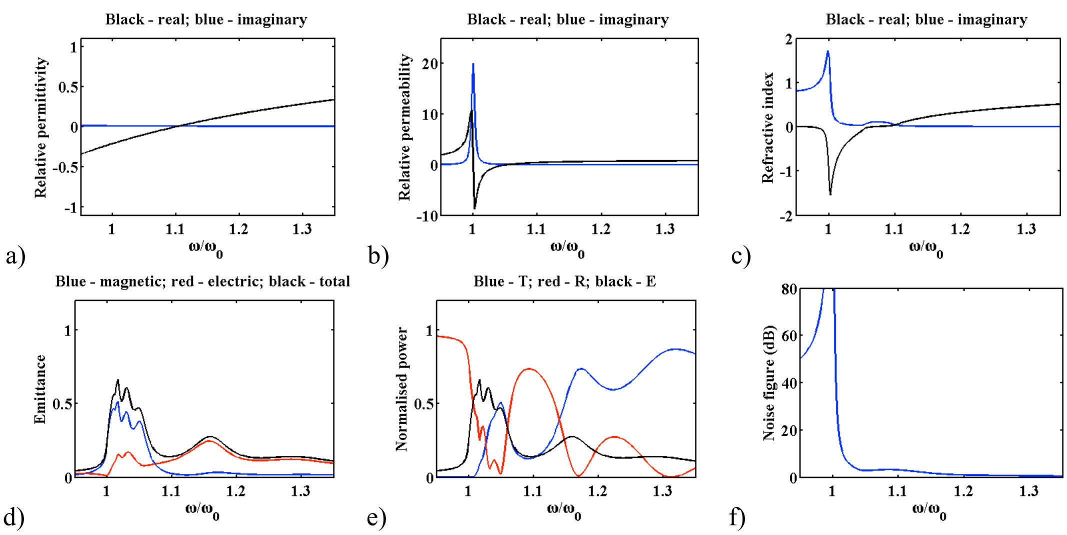

Fig. 3. Frequency variations of a) relative permittivity, b) relative permeability, c) refractive index, d) emittance, e) transmittance, reflectance and absorbance, and f) noise figure for a negative index slab.

4. Conclusions

We have demonstrated a simple transmission line calculator for electromagnetic noise, and its application to negative index media based on rings and rods. The results show that the noise figure increases rapidly in the range where n is negative due to magnetic noise from lossy resonators.

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