above, we point out, in connection with this curve, that the characteristic scale for changes in all the times in this region is found to be T, rather than ε_F, as a result of the Pauli exclusion principle.

For $\varepsilon - \varepsilon_F \gg T$, the region of small-angle scattering and that of spontaneous transitions overlap, and eqs. (4.75), (4.76) and (4.77) are transformed into eqs. (4.72), (4.74) and (4.73), respectively. This can be confirmed by means of the following asymptotic forms:

$$\mathscr{F}_n^+(0) = \frac{2^{n+1}-1}{2^{n-1}} \Gamma(n+1)\, \zeta(n+1), \tag{4.79}$$

$$\mathscr{F}_n^{\pm}(x) = \frac{x^{n+1}}{n+1} \quad \text{for } x \gg 1, \tag{4.80}$$

$$\mathscr{F}_n^-(x) = 2(1-2^{1-n})\Gamma(n+1)\, \zeta(n)\, x \quad \text{for } x \ll 1. \tag{4.81}$$

Here $\zeta(x)$ is the ζ-function.

Now we can follow the way in which the scattering changes as the electron cools if the temperature is low enough, i.e. $T \ll \hbar k_F s$. As long as $\varepsilon - \varepsilon_F \gg 2\hbar k_F s$, the predominant process is spontaneous phonon emission, on which, moreover, the Pauli exclusion principle imposes no restrictions. Since in this energy range the energy of the electron changes only slightly (because $\varepsilon - \varepsilon_F \ll \varepsilon_F$), the same set of phonons is emitted, independently of $\varepsilon - \varepsilon_F$, and therefore τ_0, τ_1 and Q are independent of $\varepsilon - \varepsilon_F$. When $\varepsilon - \varepsilon_F$ becomes smaller than $2\hbar k_F s$, the Pauli exclusion principle begins to restrict emission, and the rate of scattering drops. This decrease in scattering continues until $\varepsilon - \varepsilon_F$ becomes equal to T. In the region with $\varepsilon - \varepsilon_F \lesssim T$, the scattering time depends only weakly on energy, because scattering is by phonons with $q \sim q_T$, independently of the value of $\varepsilon - \varepsilon_F$.

4.5. *Fluctuation–dissipation theorem for quasi-elastic scattering*

In order to prove eq. (4.40), we introduce the scattering probability $W(\varepsilon \to \varepsilon')$, obtained from $W_{k \to k'}$ by averaging over the constant-energy surfaces on which $\boldsymbol{k}$ and $\boldsymbol{k}'$ lie. According to eq. (3.8), the phonon emission probability can be represented as

$$W^+(\varepsilon \to \varepsilon - \hbar\omega) = (N_\omega + 1)\, w(\varepsilon, \omega). \tag{4.82}$$

It follows, then, from the principle of detailed balance, eq. (2.10), that the absorption probability has the form

$$W^-(\varepsilon \to \varepsilon + \hbar\omega) = N_\omega w(\varepsilon + \hbar\omega, \omega). \tag{4.83}$$

Here N_ω is the Planck function and w is the probability of spontaneous emission. We also introduce the weighted quantities

$$\tilde{w}(\varepsilon, \omega) = g(\varepsilon)\ w(\varepsilon, \omega)\ g(\varepsilon - \hbar\omega), \tag{4.84}$$

$$\tilde{Q}(\varepsilon) = g(\varepsilon)\ Q(\varepsilon) \quad \text{and} \quad \tilde{D}(\varepsilon) = g(\varepsilon)\ D(\varepsilon). \tag{4.85}$$

Then eq. (4.40) assumes the simpler form

$$\tilde{Q}(\varepsilon) - \tilde{Q}^0(\varepsilon) = \tilde{Q}^T(\varepsilon) = -\partial\tilde{D}(\varepsilon)/\partial\varepsilon, \tag{4.86}$$

with

$$\tilde{Q}(\varepsilon) = \int_0^\infty \mathrm{d}(\hbar\omega)\ \hbar\omega\big[(N_\omega + 1)\tilde{w}(\varepsilon, \omega) - N_\omega\tilde{w}(\varepsilon + \hbar\omega, \omega)\big], \tag{4.87}$$

$$\tilde{D}(\varepsilon) = \tfrac{1}{2}\int_0^\infty \mathrm{d}(\hbar\omega)\ (\hbar\omega)^2\big[(N_\omega + 1)\tilde{w}(\varepsilon, \omega) + N_\omega\tilde{w}(\varepsilon + \hbar\omega, \omega)\big]. \tag{4.88}$$

In calculating $\tilde{D}$ we can disregard the difference between $\tilde{w}(\varepsilon, \omega)$ and $\tilde{w}(\varepsilon + \hbar\omega, \omega)$, and rewrite the expression within the brackets in the form $(2N_\omega + 1)\tilde{w}(\varepsilon, \omega)$. Differentiating with respect to ε, we can put

$$\frac{\partial}{\partial\varepsilon}\tilde{w}(\varepsilon, \omega) = \frac{1}{\hbar\omega}\big[\tilde{w}(\varepsilon + \hbar\omega, \omega) - \tilde{w}(\varepsilon, \omega)\big]. \tag{4.89}$$

After this, as can be readily seen, we have

$$\tilde{Q}(\varepsilon) + \frac{\partial}{\partial\varepsilon}\tilde{D}(\varepsilon) = \int_0^\infty \mathrm{d}(\hbar\omega)\ \hbar\omega\big[\tilde{w}(\varepsilon, \omega) + \tilde{w}(\varepsilon + \hbar\omega, \omega)\big]. \tag{4.90}$$

Here again we can neglect the difference between the two terms in the brackets, after which the right-hand side transforms to $\tilde{Q}^0(\varepsilon)$, which is what proves eq. (4.86).

After substituting eq. (4.40) into eq. (2.93), it becomes clear that the dynamic friction coefficient $A(\varepsilon)$, included in the expression for the current along the energy axis, eq. (2.92), is, in the case of quasi-elastic scattering by a phonon thermal bath, simply the spontaneous loss rate. Thus

$$A(\varepsilon) = Q^0(\varepsilon). \tag{4.91}$$

On the other hand, if we also make use of eq. (2.95), the total loss rate can be expressed in terms of the diffusion coefficient:

$$Q(\varepsilon) = -D(\varepsilon)\frac{\partial}{\partial\varepsilon}\ln\big[\mathrm{e}^{-\varepsilon/T}g(\varepsilon)\ D(\varepsilon)\big]. \tag{4.92}$$

This relationship between the relaxation and fluctuation characteristics remind one of the fluctuation–dissipation theorem (the Einstein relation and the Nyquist theorem).